

# Five-brane Calibrations and Fuzzy Funnels

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## Abstract

We present a generalisation of the Basu-Harvey equation that describes membranes ending on intersecting five-brane configurations corresponding to various calibrated geometries.

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# 1 Introduction

There are many different but physically equivalent descriptions of how a D1 brane may end on a D3 brane. From the point of view of the D3 brane the configuration is described by a monopole on its world volume. From the point of view of the D1 brane the configuration is described by the D1 opening up into a D3 brane where the extra two dimensions form a fuzzy two sphere whose radius diverges at the origin of the three-brane. These different view points are the stringy realisation of the Nahm transformation. The BPS equation obeyed by the D1 brane is Nahm's equation. These differing perspectives on the D1, D3 system have been explored in a variety of papers [1–6] where the fluctuation properties, the profile and the coupling to RR fields are examined and shown to match where the different approximations schemes are both valid.

The M-theory equivalent of this system is that of the membrane ending on the M five-brane. There are well known problems though of describing the theory of coincident branes in M-theory, both for the membrane and the five-brane. It is believed that matrix valued fields will not be the appropriate degrees of freedom as they are for D-branes since entropy considerations imply a different number of degrees of freedom than one would expect from matrix valued fields.

What is known is the five-brane equivalent of the BIon solution. This is the self-dual string solution of Howe, Lambert and West [7] whose properties have recently been investigated in [8]. The absence of a non-Abelian membrane theory however meant that the equivalent of Nahm's equation for the self-dual string was missing. Recently, Basu and Harvey [9], made an ansatz for such an equation. Their goal being to produce a similar fuzzy funnel description of the membrane opening up into the five-brane. From their generalised Nahm equation they went on to infer (essentially through an inverse Bogomol'nyi argument) a non-Abelian membrane action with sextic interaction. Again, the profile and the fluctuations were shown to agree with the self-dual string description.

Given the somewhat ad-hoc way in which the Basu-Harvey equation has been determined it would be good to see if one could find other tests for its validity. The goal of this paper will be to show that the Basu-Harvey equation with a natural generalisation can describe not only the membrane ending on a five-brane but also the membrane ending on various intersecting five-brane configurations. These five-brane configurations will correspond to calibrated cycles. Essentially this is the M-theory generalisation of [10] where

the Nahm equation was generalised to describe the D1 ending on the configurations of D3 branes that correspond to three-branes wrapped on calibrated cycles.

We proceed by first describing Nahm's equation and the generalisation that leads to the description of a D1 ending on intersecting D3-brane configurations. As an aside we also demonstrate that the solutions previously derived using the linearised approximation in [10] are actually solutions of the full non-linear theory. This is undoubtedly a consequence of the BPS nature of these solutions. Then we discuss the M-theory generalisation of Nahm's equation as introduced by Basu and Harvey. Finally we describe a generalisation of the Basu-Harvey equation and its solutions that correspond to membranes ending on five-branes wrapped on calibrated cycles.

## 2 Nahm Type Equations

The Nahm equation [13] is given by

$$\frac{\partial \Phi^i}{\partial x^9} = \pm \frac{i}{2} \epsilon_{ijk} [\Phi^j, \Phi^k]. \quad (1)$$

This equation is derived in string theory by simply examining the  $\frac{1}{2}$  BPS equation for the D1 brane [14, 15]. Its solutions correspond to D1 branes opening up into a fuzzy funnel to form a D3 brane. Note, the boundary conditions are taken such that  $\Phi^i(x^9)$  is defined over the semi-infinite line as opposed to a finite interval which is the usual case. This corresponds to infinite mass monopoles which have the interpretation of infinite D1 strings ending on the brane. Explicitly, the solutions are:

$$\Phi^i = \pm \frac{1}{2(\sigma - \sigma_0)} \alpha^i. \quad (2)$$

Where  $\alpha^i$  obey the  $su(2)$  algebra,  $[\alpha^i, \alpha^j] = 2i\epsilon^{ijk}\alpha^k$ . The sign choice is related to whether the solution is BPS or anti-BPS, in what follows a particular sign will be chosen though one should keep in mind that one can choose the opposite sign and the resulting solutions will simply be the anti-BPS equivalent. Nahm's equation was generalised in [10] by considering not just the  $\frac{1}{2}$  BPS equation of the D1 brane but instead by looking at the BPS equation that arises from preserving a lower number of supersymmetries. This produced the following generalised Nahm equation:

$$\frac{\partial \Phi^i}{\partial x^9} = \pm \frac{i}{2} c_{ijk} [\Phi^j, \Phi^k], \quad (3)$$

where  $c_{ijk}$  is some totally anti-symmetric constant tensor with  $i, j, k = 1, \dots, d$ . Along with this Nahm type equation there is also a set of algebraic equations that arise from the BPS conditions. This generalised Nahm equation along with the algebraic equations together imply the equations of motion. (We will not give the algebraic equations here since they are dependent on the details of the preserved supersymmetry, see [10] for details).

Spinors that obey the supersymmetry projection conditions may be used to write down a calibration form,  $c = c_{ijk} dx^i \wedge dx^j \wedge dx^k$  using  $c_{ijk} = \bar{\epsilon} \Gamma_{ijk} \epsilon$ . The components of this form  $c_{ijk}$  are then what appear in the modified Nahm equation given above. (It should be stated that this modified Nahm equation with a specific  $c$  has appeared in the pre D-brane literature, for example [16]). The solutions to this equation then correspond to D1 branes ending on a three-brane that wraps the calibrated cycle or equivalently a set of intersecting three-branes. The description of the relation between calibrated cycles and intersecting brane configurations in string theory was described in [17, 18].

The analysis performed in [10] examined the linearised D1 brane action. Here we examine the full non-Abelian Born-Infeld action for the D1, given by

$$S = -T_1 \int d^2\sigma STr \sqrt{-\det(\eta_{ab} + \lambda^2 \partial_a \Phi^i Q_{ij}^{-1} \partial_b \Phi^j) \det(Q^{ij})}, \quad (4)$$

where  $Q^{ij} = \delta^{ij} + i\lambda[\Phi^i, \Phi^j]$  and  $\lambda = 2\pi l_s^2$ . STr denotes the symmetrised trace prescription [19]. In 2-dimensions the gauge field carries no propagating degrees of freedom and may be completely gauged away, which is why only partial derivatives appear in the above action. (It is known that there is some possible ambiguity in the non-Abelian Born-Infeld theory since the derivative approximation is not valid in a non-abelian theory yet this action has been shown to possess many of the right properties, see for example [3, 4]).

We now wish to write down the energy so as to obtain a Bogomol'nyi style argument. If we restrict ourselves to a static solution with three non-zero scalars, which only depend on  $\Phi^9$  we can expand the determinant to give an expression for the energy

$$E = T_1 \int d\sigma STr \sqrt{I + \lambda^2 \partial \Phi^i \partial \Phi^i - \frac{1}{2} \lambda^2 [\Phi^i, \Phi^j]^2 - \left( \frac{1}{2} \lambda^2 \epsilon_{ijk} \partial \Phi^i [\Phi^j, \Phi^k] \right)^2}, \quad (5)$$

the terms under the square root can then be rewritten using the Nahm equation as a perfect square so that

$$E = T_1 \int d\sigma STr \left( I + \frac{1}{2} \lambda^2 \left( \partial \Phi^i \partial \Phi^i - [\Phi^i, \Phi^j]^2 \right) \right), \quad (6)$$

and we can clearly see that energy reduces to the linear form and a solution to the Nahm equation solves the full non-linear theory.

It was from this linearised starting point that [10] proceeded by using the Bogomol'nyi trick to write the energy (minus the constant piece) as

$$E = T_1 \int d\sigma STr \left( \frac{1}{2} \lambda^2 \left( \partial \Phi^i - \frac{1}{2} c_{ijk} [\Phi^j, \Phi^k] \right)^2 + T \right) \quad (7)$$

where  $T$  is a topological piece. In order to be able to write the energy in this form one requires that

$$\frac{1}{2} c_{ijk} c_{ilm} Tr ([\Phi^j, \Phi^k] [\Phi^l, \Phi^m]) = Tr ([\Phi^i, \Phi^j] [\Phi^i, \Phi^j]) . \quad (8)$$

One can then show that the above equation (8) holds along with the equations of motion derived from the action (4) if the modified Nahm equation is obeyed along with the algebraic conditions that follow from the vanishing of the supersymmetry variation. The modified Nahm equation then manifestly appears as the Bogomol'nyi equation derived by minimising (7).

The first configuration considered in [10] is with two intersecting three-branes. This configuration requires (3) to be satisfied with  $c_{123} = c_{145} = 1$  as well as the associated algebraic equations that follow from the supersymmetry. Expanding the energy for five non-zero scalars we have

$$E = T_1 \int d^2\sigma STr \sqrt{\left( I + \frac{1}{2} \lambda^2 (\partial \Phi^i \partial \Phi^i - \frac{1}{2} [\Phi^i, \Phi^j]^2) \right)^2 + \lambda^6 (\epsilon_{ijklm} \partial \Phi^i [\Phi^j, \Phi^k] [\Phi^l, \Phi^m])^2} \quad (9)$$

where we have used the modified Nahm equation and associated algebraic conditions to write the first square in that form. Thus if the epsilon term vanishes for a solution to the linear equations of motion, it is also solution to the full non-linear equations of motion. Indeed one can check that for the solutions given in [10] that this is the case and so their solutions are again solutions of the full Born-Infeld theory. It is interesting to note however that there do exist solutions to (3) which do not have the form described in [10] where this second term in the energy does not vanish. These solutions correspond to the case where the calibration is deformed away from the flat intersection [20, 21]. The nonlinearity of the brane action then plays a key role. In what follows we will restrict ourselves to the case of flat intersecting branes so that the linear equations will be enough. It would be interesting for future work to examine the case where the calibration is deformed away from the simple intersection, from the membrane perspective.

In summary, the non-linear action is such that solutions that saturate the bound of the linear theory are also solutions of the non-linear theory. In the case of less than half supersymmetry there are algebraic equations in addition to the Nahm type equation. These algebraic equations are actually necessary to derive a Bogomol'nyi bound. These are the guiding properties that we will use to extend the Basu-Harvey equation and the associated membrane action to describe more complicated five-brane configurations.

### 3 A Nahm Equation for M-theory

The following M-theory version of the Nahm equation was proposed in [9], to describe membranes ending on a five-brane,

$$\frac{\partial X^i}{\partial s} + \frac{M_{11}^3}{64\pi} \epsilon_{ijkl} \frac{1}{4!} [G_5, X^j, X^k, X^l] = 0, \quad (10)$$

where the quantum Nambu 4-bracket is defined by

$$[A_1, A_2, A_3, A_4] = \sum_{\text{permutations } \sigma} \text{sgn}(\sigma) A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} A_{\sigma_4} \quad (11)$$

and  $G_5$  is a difference of projection operators defined in the Appendix. The solution to this equation as given in [9] is

$$X^i(s) = \frac{i\sqrt{2\pi}}{M_{11}^{3/2}} \frac{1}{\sqrt{s}} G^i, \quad (12)$$

where the set of matrices  $\{G^i\}$  are in a particular representation of  $\text{Spin}(4)$  (see Appendix). Thus, the solution is again that of a fuzzy funnel but this time there is a fuzzy three-sphere whose radius diverges to form the five-brane.

Just as the Nahm equation was generalised to describe D1 branes ending on a three-brane that wraps a calibrated cycle so we wish to modify the Basu-Harvey equation to describe membranes ending on a five-brane that wraps some calibrated cycle. The ability of the Basu-Harvey equations to be modified in a natural way so as to allow these more general configurations will be taken as supporting evidence in favour of the validity of their equation in describing membranes.

The natural generalisation of the Basu-Harvey equation which we propose is

$$\frac{\partial X^i}{\partial s} + \frac{M_{11}^3}{64\pi} g_{ijkl} \frac{1}{4!} [H^*, X^j, X^k, X^l] = 0, \quad (13)$$

where  $g_{ijkl}$  is an anti-symmetric constant tensor that is associated to the components of the relevant calibration form that represents an intersecting five-brane configuration.  $H^*$  has

properties analogous to those of  $G_5$ , namely  $(H^*)^2 = 1$  and for the solutions we consider  $\{H^*, X^i\} = 0$ .

In [9] an expression for the membrane energy in such a configuration was also postulated,

$$E = T_2 \int d^2\sigma Tr \left[ \left( X^{i'} + \epsilon_{ijkl} \frac{1}{4!} [G_5, X^j, X^k, X^l] \right)^2 + \left( 1 - \frac{1}{2} \epsilon_{ijkl} \{X^{i'}, \frac{1}{4!} [G_5, X^j, X^k, X^l]\} \right)^2 \right]^{1/2}. \quad (14)$$

Now the  $G_5$ 's drop out of this expression, and using the Basu-Harvey equation it can be rewritten as

$$E = T_2 \int d^2\sigma Tr \left[ \left( 1 + \frac{1}{2} (\partial_a X^i)^2 - \frac{1}{2 \cdot 3!} [X^i, X^j, X^k]^2 \right)^2 \right]^{1/2} \quad (15)$$

where  $[X^i, X^j, X^k]$  is a Nambu 3-bracket containing all permutations of the three entries with signs. So for three active scalars if the Basu-Harvey equation is obeyed the action proposed in [9] is equivalent to its *linearised* version

$$S = -T_2 \int d^3\sigma Tr \left( 1 + \frac{1}{2} (\partial_a X^i)^2 - \frac{1}{2 \cdot 3!} [X^j, X^k, X^l]^2 \right). \quad (16)$$

We propose that when more scalars are activated we can use our modified Basu-Harvey equation (13) to rewrite the action as the linear piece squared plus other squared terms. When these other terms are zero, the *linearised* action is the full action. The energy is then given by

$$E = \frac{T_2}{2} \int d^2\sigma Tr \left( X^{i'} X^{i'} - \frac{1}{3!} [X^j, X^k, X^l] [X^j, X^k, X^l] \right) \quad (17)$$

where we have subtracted the constant piece and assumed that the  $X^i$  depend only on  $\sigma_2 (= X^{10}$  in this gauge). Following the usual Bogomol'nyi construction we rewrite this as

$$E = \frac{T_2}{2} \int d^2\sigma \left\{ Tr \left( X^{i'} + g_{ijkl} \frac{1}{4!} [H^*, X^j, X^k, X^l] \right)^2 + T \right\} \quad (18)$$

where  $T$  is a *topological* piece given by

$$T = -T_2 \int d^2\sigma Tr \left( \frac{M_{11}^3}{64\pi} g_{ijkl} \frac{\partial X^i}{\partial \sigma_2} \frac{1}{4!} [H^*, X^j, X^k, X^l] \right) \quad (19)$$

(with factors restored). This reproduces the correct energy density for the five-brane on which the membranes end in the case where  $g_{ijkl} = \epsilon_{ijkl}$ . Now, to rewrite in the energy in

this way we have imposed that

$$\begin{aligned} & \frac{1}{3!} g_{ijkl} g_{ipqr} \text{Tr} ([H^*, X^j, X^k, X^l][H^*, X^p, X^q, X^r]) \\ &= \text{Tr} ([H^*, X^i, X^j, X^k][H^*, X^i, X^j, X^k]) \end{aligned} \quad (20)$$

which using  $\{H^*, X^i\} = 0$  and  $(H^*)^2 = 1$  is equivalent to the following algebraic constraint

$$\frac{1}{3!} g_{ijkl} g_{ipqr} \text{Tr} ([X^j, X^k, X^l][X^p, X^q, X^r]) = \text{Tr} ([X^i, X^j, X^k][X^i, X^j, X^k]) . \quad (21)$$

Note, this is satisfied for the case  $g_{ijkl} = \epsilon_{ijkl}$  when the only scalars activated are  $X^2$  to  $X^5$ . The Bogomol'nyi equation found from minimising the energy given in equation (18) is then our modified Basu-Harvey equation (13).

We now show explicitly that the equation of motion following from the action (16) when combined with the modified Basu-Harvey equation imply the constraint (21).

The equation of motion is given by

$$X^{i''} = -\frac{1}{2} [X^j, X^k, [X^i, X^j, X^k]] \quad (22)$$

where the three bracket  $[A, B, C]$  is the sum of the six permutations of the three entries, but with the sign of the permutation determined only by the order of the first two entries, i.e.  $ABC, ACB$  and  $CAB$  are the positive permutations. By using the Bogomol'nyi equation (13) twice on the left-hand side we get:

$$\frac{1}{3!} g_{ijkl} g_{jpqr} [X^k, X^l, [X^p, X^q, X^r]] = -[X^j, X^k, [X^i, X^j, X^k]] . \quad (23)$$

After multiplying by  $X^i$  and taking the trace, we recover the above constraint equation, (21). Thus in summary, the solutions of the generalised Basu-Harvey equation (13) that obey the constraint equation (23) are solutions to the equations of motion of the proposed membrane action (16).

## 4 Supersymmetry

In the D-brane case, both the Nahm like equation and the additional algebraic relations could be derived from imposing that the necessary supersymmetry variation vanished. The approach we have described above is equivalent but less efficient. The Bogomol'nyi argument effectively implies the Nahm type equation and the necessary algebraic relations. We would like to encode this information by imposing by fiat a supersymmetry transformation whose vanishing will imply the above equations. As to whether this can be made



more concrete by constructing a supersymmetric membrane action with this supersymmetry variation we leave as an interesting open question. The advantage of this imposed supersymmetry variation is that it will allow us to relate the solutions of the membrane equations to intersecting five-branes that preserve various fractions of supersymmetry. The obvious suggested susy variation is

$$\delta\lambda = \left( \frac{1}{2} \partial_\mu X^i \Gamma^{\mu i} - \frac{1}{2 \cdot 4!} [H^*, X^i, X^j, X^k] \Gamma^{ijk} \right) \epsilon. \quad (24)$$

Now, we use the modified Basu-Harvey equation in the first term and rearrange so that the requirement that the supersymmetry variation vanishes becomes that

$$\sum_{i < j < k} [X^i, X^j, X^k] \Gamma^{ijk} (1 - g_{ijkl} \Gamma^{ijkl\#}) \epsilon = 0, \quad (25)$$

we have removed and overall factor of  $H^*$  from the left-hand side since, like  $G_5$ , it is the difference of projection operators onto orthogonal sub-spaces and has trivial kernel.  $\epsilon$  is the preserved supersymmetry on the membrane worldvolume and we have  $\Gamma^{01\#} \epsilon = \epsilon$  where the membrane's worldvolume is in the 0, 1 and  $10 = \#$  directions. We can then solve the supersymmetry condition (25) by defining projectors

$$P_{ijkl} = \frac{1}{2} (1 - g_{ijkl} \Gamma^{ijkl\#}) \quad (26)$$

where there is no sum over  $i, j, k$  or  $l$ . We normalise  $g_{ijkl} = \pm 1$  so they obey  $P_{ijkl} P_{ijkl} = P_{ijkl}$ . (Note, in all the cases that we will consider, for each triplet  $i, j, k$ ,  $g_{ijkl}$  is only non-zero for at most one value of  $l$ ). We impose  $P_{ijkl} \epsilon = 0$  for each  $i, j, k, l$  such that  $g_{ijkl} \neq 0$ . Then by using the membrane projection ( $\Gamma^{01\#} \epsilon = \epsilon$ ) we can see that each projector  $P_{ijkl}$  corresponds to a five-brane in the 0, 1,  $i, j, k, l$  directions. To apply the projectors simultaneously, the matrices  $\Gamma_{ijkl\#}$  need to commute with each other.  $[\Gamma_{ijkl\#}, \Gamma_{i'j'k'l'\#}] = 0$  if and only if the sets  $\{i, j, k, l\}$  and  $\{i', j', k', l'\}$  have two or zero elements in common, corresponding to five-branes intersecting over a three-brane soliton or a string soliton.

Once we impose the set of mutually commuting projectors, the supersymmetry transformation (25) reduces to

$$\sum_{g_{ijkl}=0} [X^i, X^j, X^k] \Gamma^{ijk} \epsilon = 0. \quad (27)$$

Here we sum over triplets  $i, j, k$ , such that  $g_{ijkl} = 0$  for all  $l$ . Using the projectors allows us to express these as a set of conditions on the 3-brackets alone.

## 5 Five-Brane Configurations

We will now describe the specific equations that emerge which correspond to the various possible intersecting five-brane configurations.

The five-branes must always have at least one spatial direction in common corresponding to the direction in which the membrane intersects the five-branes. These configurations of five-branes can also be thought of as a single five-brane stretched over a calibrated manifold [22]. These five-brane intersections can be found in [17, 18, 23]. We list the conditions following from the modified Basu-Harvey Equation, those following from the supersymmetry conditions (27) (with  $\nu$  the fraction of preserved supersymmetry) and then discuss any remaining conditions required to satisfy the constraint (23). In string case, [10] only the supersymmetry conditions and the Jacobi identity were required to satisfy the equivalent constraint.

### 5.1 Configuration 1

The first configuration corresponding to the single five-brane [9] is

$$\begin{array}{lllll}
M5: & 1 & 2 & 3 & 4 & 5 \\
M2: & 1 & & & & \# \\
& & g_{2345} = 1 & & \nu = 1/2 & 
\end{array} \quad (28)$$

$$\begin{aligned} X^{2'} &= -H^*[X^3, X^4, X^5] \quad , & X^{3'} &= H^*[X^4, X^5, X^2] \quad , \\ X^{4'} &= -H^*[X^5, X^2, X^3] \quad , & X^{5'} &= H^*[X^2, X^3, X^4] \quad . \end{aligned}$$

## 5.2 Configuration 2

For the next case we consider two five-branes intersecting on a three-brane corresponding to an  $SU(2)$  Kahler calibration of a two surface embedded in four dimensions. The activated scalars are  $X^2$  to  $X^7$ .

$$\begin{array}{cccccccc}
M5 : & 1 & 2 & 3 & 4 & 5 & & \\
M5 : & 1 & 2 & 3 & & & 6 & 7 \\
M2 : & 1 & & & & & & \# \\
g_{2345} = g_{2367} = 1 & & & & & & \nu = 1/4 & (29)
\end{array}$$

$$\begin{aligned}
X^{2'} &= -H^*[X^3, X^4, X^5] - H^*[X^3, X^6, X^7] \quad , \quad X^{3'} = H^*[X^4, X^5, X^2] + H^*[X^6, X^7, X^2] \quad , \\
X^{4'} &= -H^*[X^5, X^2, X^3] \quad , \quad X^{5'} = H^*[X^2, X^3, X^4] \quad , \\
X^{6'} &= -H^*[X^7, X^2, X^3] \quad , \quad X^{7'} = H^*[X^2, X^3, X^6] \quad , \\
[X^2, X^4, X^6] &= [X^2, X^5, X^7] \quad , \quad [X^2, X^5, X^6] = -[X^2, X^4, X^7] \quad , \\
[X^3, X^4, X^6] &= [X^3, X^5, X^7] \quad , \quad [X^3, X^5, X^6] = -[X^3, X^4, X^7] \quad , \\
[X^4, X^5, X^6] &= [X^4, X^5, X^7] = [X^4, X^6, X^7] = [X^5, X^6, X^7] = 0.
\end{aligned}$$

In order to satisfy the constraint we need the  $X^i$ 's to satisfy the following equations:

$$\text{Choose } m \in \{2, 3\}, \quad i, j, k, l \in \{4, 5, 6, 7\}$$

$$\begin{aligned}
\epsilon_{ijk}[X^i, X^m, [X^m, X^j, X^k]] &= 0, \quad (\text{no sum over } m) \\
\epsilon_{ijkl}[X^i, X^j, [X^m, X^k, X^l]] &= 0.
\end{aligned} \tag{30}$$

In the string theory case there were no additional equations, as apart from the Nahm like equations and algebraic conditions on the brackets all that was needed to solve the constraint was the Jacobi identity. If  $X^m$  anti-commutes with  $X^i, X^j, X^k$  then the first equation reduces to the Jacobi identity. Similarly if  $X^m$  anti-commutes with  $X^i, X^j, X^k, X^l$  the second equation reduces to

$$\epsilon_{ijkl}X^iX^jX^kX^l = 0. \tag{31}$$

### 5.3 Configuration 3

Three five-branes can intersect on a three-brane corresponding to an  $SU(3)$  Kahler calibration of a two surface embedded in six dimensions. The active scalars are  $X^2$  to  $X^9$ .

$$\begin{aligned}
M5 : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
M5 : & \quad 1 \quad 2 \quad 3 \quad \quad \quad 6 \quad 7 \\
M5 : & \quad 1 \quad 2 \quad 3 \quad \quad \quad \quad 8 \quad 9 \\
M2 : & \quad 1 \quad \quad \quad \quad \quad \quad \quad \quad \# \\
g_{2345} = g_{2367} = g_{2389} &= 1 \quad \quad \quad \nu = 1/8
\end{aligned} \tag{32}$$

$$\begin{aligned}
X^{2'} &= -H^*[X^3, X^4, X^5] - H^*[X^3, X^6, X^7] - H^*[X^3, X^8, X^9], \\
X^{3'} &= H^*[X^4, X^5, X^2] + H^*[X^6, X^7, X^2] + H^*[X^8, X^9, X^2],
\end{aligned}$$

$$\begin{aligned}
X^{4'} &= -H^*[X^5, X^2, X^3] \quad , \quad X^{5'} = H^*[X^2, X^3, X^4] \quad , \\
X^{6'} &= -H^*[X^7, X^2, X^3] \quad , \quad X^{7'} = H^*[X^2, X^3, X^6] \quad , \\
X^{8'} &= -H^*[X^9, X^2, X^3] \quad , \quad X^{9'} = H^*[X^2, X^3, X^8] \quad , \\
[X^2, X^4, X^6] &= [X^2, X^5, X^7] \quad , \quad [X^2, X^5, X^6] = -[X^2, X^4, X^7], \\
[X^2, X^4, X^8] &= [X^2, X^5, X^9] \quad , \quad [X^2, X^5, X^8] = -[X^2, X^4, X^9] \quad , \\
[X^2, X^6, X^8] &= [X^2, X^7, X^9] \quad , \quad [X^2, X^7, X^8] = -[X^2, X^6, X^9], \\
[X^3, X^4, X^6] &= [X^3, X^5, X^7] \quad , \quad [X^3, X^5, X^6] = -[X^3, X^4, X^7], \\
[X^3, X^4, X^8] &= [X^3, X^5, X^9] \quad , \quad [X^3, X^5, X^8] = -[X^3, X^4, X^9] \quad , \\
[X^3, X^6, X^8] &= [X^3, X^7, X^9] \quad , \quad [X^3, X^7, X^8] = -[X^3, X^6, X^9], \\
[X^4, X^5, X^6] + [X^6, X^8, X^9] &= 0 \quad , \quad [X^4, X^5, X^7] + [X^7, X^8, X^9] = 0 \quad , \\
[X^4, X^5, X^8] + [X^6, X^7, X^8] &= 0 \quad , \quad [X^4, X^5, X^9] + [X^6, X^7, X^9] = 0 \quad , \\
[X^4, X^6, X^7] + [X^4, X^8, X^9] &= 0 \quad , \quad [X^5, X^6, X^7] + [X^5, X^8, X^9] = 0 \quad , \\
[X^4, X^6, X^8] &= [X^4, X^7, X^9] \quad + \quad [X^5, X^6, X^9] + [X^5, X^7, X^8] \quad , \\
[X^5, X^7, X^9] &= [X^5, X^6, X^8] \quad + \quad [X^4, X^7, X^8] + [X^4, X^6, X^9] \quad .
\end{aligned}$$

In order to satisfy the constraint again we have additional algebraic constraints for certain  $X^i$ 's, for this we define “pairs” as  $\{2, 3\}, \{4, 5\}, \{6, 7\}$  and  $\{8, 9\}$

Choose  $m \in \{2, 3\}$ ,  $i, j, k, l \in \{4, 5, 6, 7, 8, 9\}$  such that  $\{i, j\}, \{k, l\}$  are pairs

$$\begin{aligned}
\epsilon_{ijk}[X^i, X^m, [X^m, X^j, X^k]] &= 0, \quad (\text{no sum over } m) \\
\epsilon_{ijkl}[X^i, X^j, [X^m, X^k, X^l]] &= 0.
\end{aligned} \tag{33}$$

## 5.4 Configuration 4

The next configuration has 3 five-branes intersecting over a string which corresponds to an SU(3) Kahler calibration of a four surface in six dimensions. There are only 6 activated scalars.

$$\begin{array}{cccccc}
M5 : & 1 & 2 & 3 & 4 & 5 \\
M5 : & 1 & 2 & 3 & & 6 & 7 \\
M5 : & 1 & & & 4 & 5 & 6 & 7 \\
M2 : & 1 & & & & & & \# \\
g_{2345} = g_{2367} = g_{4567} &= 1 & & & & \nu = 1/8
\end{array} \tag{34}$$

$$\begin{aligned}
X^{2'} &= -H^*[X^3, X^4, X^5] - H^*[X^3, X^6, X^7] , \\
X^{3'} &= H^*[X^2, X^4, X^5] + H^*[X^2, X^6, X^7] , \\
X^{4'} &= -H^*[X^2, X^3, X^5] - H^*[X^5, X^6, X^7] , \\
X^{5'} &= H^*[X^2, X^3, X^4] + H^*[X^4, X^6, X^7] , \\
X^{6'} &= -H^*[X^2, X^3, X^7] - H^*[X^4, X^5, X^7] , \\
X^{7'} &= H^*[X^2, X^3, X^6] + H^*[X^4, X^5, X^6] , \\
[X^2, X^4, X^6] &= [X^2, X^5, X^7] + [X^3, X^5, X^6] + [X^3, X^4, X^7] , \\
[X^3, X^5, X^7] &= [X^3, X^4, X^6] + [X^2, X^5, X^6] + [X^2, X^4, X^7] .
\end{aligned}$$

In order to satisfy the constraint again we have to satisfy additional algebraic constraints for certain  $X^i$ 's, for this we define “pairs” as  $\{2, 3\}, \{4, 5\}$  and  $\{6, 7\}$

Choose  $i, j, k, l, m \in \{2, 3, 4, 5, 6, 7\}$  such that  $\{i, j\}, \{k, l\}$  are pairs

$$\begin{aligned}
\epsilon_{ijk}[X^i, X^m, [X^m, X^j, X^k]] &= 0, \quad (\text{no sum over } m) \\
\epsilon_{ijkl}[X^i, X^j, [X^m, X^k, X^l]] &= 0.
\end{aligned} \tag{35}$$

## 5.5 Configuration 5

In the next configuration we are forced by supersymmetry to have an additional anti-brane. Even though there are only three independent projectors this configuration has three five-branes and an anti-five-brane intersecting over a membrane. This corresponds to the  $SU(3)$  special Lagrangian calibration of a three surface embedded in six dimensions.

$$\begin{array}{llllll}
M5: & 1 & 2 & 3 & 4 & 5 \\
M5: & 1 & 2 & & 4 & 6 & 8 \\
\bar{M}5: & 1 & 2 & 3 & & 6 & 7 \\
M5: & 1 & 2 & & 5 & 7 & 8 \\
M2: & 1 & & & & & \# \\
g_{2345} = g_{2468} = -g_{2367} = g_{2578} = 1 & & & & & \nu = 1/8
\end{array} \tag{36}$$

$$\begin{aligned}
X^{2'} &= -H^*[X^3, X^4, X^5] - H^*[X^4, X^6, X^8] + H^*[X^3, X^6, X^7] - H^*[X^5, X^7, X^8] , \\
X^{3'} &= H^*[X^2, X^4, X^5] - H^*[X^2, X^6, X^7] , \\
X^{4'} &= -H^*[X^2, X^3, X^5] - H^*[X^2, X^6, X^8] ,
\end{aligned}$$

$$\begin{aligned}
X^{5'} &= H^*[X^2, X^3, X^4] + H^*[X^2, X^7, X^8] , \\
X^{6'} &= H^*[X^2, X^3, X^7] - H^*[X^2, X^4, X^8] , \\
X^{7'} &= -H^*[X^2, X^3, X^6] - H^*[X^2, X^5, X^8] , \\
X^{8'} &= H^*[X^2, X^4, X^6] + H^*[X^2, X^5, X^7] , \\
[X^3, X^4, X^7] &= [X^3, X^5, X^6] , \quad [X^4, X^3, X^8] = [X^4, X^5, X^6] , \\
[X^5, X^3, X^8] &= [X^5, X^7, X^4] , \quad [X^6, X^4, X^7] = [X^6, X^8, X^3] , \\
[X^7, X^3, X^8] &= [X^7, X^6, X^5] , \quad [X^8, X^4, X^7] = [X^8, X^6, X^5] , \\
[X^2, X^3, X^8] + [X^2, X^4, X^7] + [X^2, X^6, X^5] &= 0 , \\
[X^6, X^7, X^8] + [X^4, X^5, X^8] + [X^3, X^4, X^6] + [X^3, X^5, X^7] &= 0 .
\end{aligned}$$

In order to satisfy the constraint once again we have similar additional algebraic constraints for certain  $X^i$ 's, for this we define “pairs” as  $\{3, 8\}, \{4, 7\}$  and  $\{5, 6\}$

Choose  $m \in \{2, \dots, 8\}$ ,  $i, j, k, l \in \{3, 4, 5, 6, 7, 8\}$  such that  $\{i, j\}, \{k, l\}$  are pairs

$$\begin{aligned}
\epsilon_{ijk}[X^i, X^m, [X^m, X^j, X^k]] &= 0, \quad (\text{no sum over } m) \\
\epsilon_{ijkl}[X^i, X^j, [X^m, X^k, X^l]] &= 0.
\end{aligned} \tag{37}$$

## 6 Solutions

The Basu-Harvey equation is solved by

$$X^i(s) = \frac{i\sqrt{2\pi}}{M_{11}^{3/2}} \left( \frac{2n+6}{2n+1} \right)^{1/2} \frac{1}{\sqrt{s}} G^i \tag{38}$$

where  $n$  labels the specific representation of  $Spin(4)$  (see the Appendix). We can solve the cases of intersecting five-branes analogously to the intersecting three-branes of [10] by effectively using multiple copies of this solution. The first multi-five-brane case (29) is solved by setting

$$X^i(s) = \frac{i\sqrt{2\pi}}{M_{11}^{3/2}} \left( \frac{2n+6}{2n+1} \right)^{1/2} \frac{1}{\sqrt{s}} H^i \tag{39}$$

where the  $H^i$  are given by the block-diagonal  $2N \times 2N$  matrices

$$\begin{aligned}
H^2 &= \text{diag}(G^1, G^1) \\
H^3 &= \text{diag}(G^2, G^2)
\end{aligned}$$

$$\begin{aligned}
H^4 &= \text{diag}(G^3, 0) \\
H^5 &= \text{diag}(G^4, 0) \\
H^6 &= \text{diag}(0, G^3) \\
H^7 &= \text{diag}(0, G^4) \\
H^* &= \text{diag}(G^5, G^5),
\end{aligned} \tag{40}$$

which are such that

$$H^i + \frac{n+3}{8(2n+1)} g_{ijkl} \frac{1}{4!} [H^*, H^j, H^k, H^l] = 0. \tag{41}$$

This makes sure that the conditions following from the generalised Basu-Harvey equation vanish. The remaining conditions in (29), that is those following from the supersymmetry transformation, are satisfied trivially as the three brackets involved all vanish for this solution. The first additional algebraic equation of (30) is satisfied for the solution as the indices must be chosen such that for each diagonal block at least one of the  $X^i$ 's appearing in the bracket that has a zero there, thus the term with each permutation vanishes independently. Again the second additional algebraic equation is trivially satisfied as there are no non-zero products of 5 different  $X^i$ 's.

The more complicated cases follow easily: configuration 3 is given by the block-diagonal  $3N \times 3N$  matrices

$$\begin{aligned}
H^2 &= \text{diag}(G^1, G^1, G^1) \\
H^3 &= \text{diag}(G^2, G^2, G^2) \\
H^4 &= \text{diag}(G^3, 0, 0) \\
H^5 &= \text{diag}(G^4, 0, 0) \\
H^6 &= \text{diag}(0, G^3, 0) \\
H^7 &= \text{diag}(0, G^4, 0) \\
H^8 &= \text{diag}(0, 0, G^3) \\
H^9 &= \text{diag}(0, 0, G^4) \\
H^* &= \text{diag}(G^5, G^5, G^5),
\end{aligned} \tag{42}$$

and configuration 4 by

$$\begin{aligned}
H^2 &= \text{diag}(G^1, G^1, 0) \\
H^3 &= \text{diag}(G^2, G^2, 0) \\
H^4 &= \text{diag}(G^3, 0, G^1) \\
H^5 &= \text{diag}(G^4, 0, G^2) \\
H^6 &= \text{diag}(0, G^3, G^3) \\
H^7 &= \text{diag}(0, G^4, G^4) \\
H^* &= \text{diag}(G^5, G^5, G^5).
\end{aligned} \tag{43}$$

Configuration 5 is

$$\begin{aligned}
H^2 &= \text{diag}(G^1, G^1, G^1, G^1) \\
H^3 &= \text{diag}(G^2, 0, G^2, 0) \\
H^4 &= \text{diag}(G^3, G^2, 0, 0) \\
H^5 &= \text{diag}(G^4, 0, 0, G^2) \\
H^6 &= \text{diag}(0, G^3, G^4, 0) \\
H^7 &= \text{diag}(0, 0, G^3, G^3) \\
H^8 &= \text{diag}(0, G^4, 0, G^4) \\
H^* &= \text{diag}(G^5, G^5, G^5, G^5).
\end{aligned} \tag{44}$$

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## A The Fuzzy 3-Sphere

To make this self-contained we include a brief description of the fuzzy 3-sphere construction following that in [9]. The approach was developed in [25–28], with an interesting string



interpretation further developed in [29].

Unlike even fuzzy spheres we must use reducible representations of  $spin(4) = SU(2) \times SU(2)$ .  $\mathcal{R}^+$  and  $\mathcal{R}^-$  are the  $(\frac{n+1}{4}, \frac{n-1}{4})$  and  $(\frac{n-1}{4}, \frac{n+1}{4})$  representations respectively where  $n$  is an odd integer. The dimension of  $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$  is given by  $N = (n+1)(n+3)/2$ .

The coordinates on the fuzzy  $S^3$  are the  $N \times N$  matrices  $G^i$  ( $i = 1$  to 4). These matrices are defined by

$$G^i = \mathcal{P}_{\mathcal{R}^+} \sum_{s=1}^n \rho_s(\Gamma^i P_-) \mathcal{P}_{\mathcal{R}^-} + \mathcal{P}_{\mathcal{R}^-} \sum_{s=1}^n \rho_s(\Gamma^i P_+) \mathcal{P}_{\mathcal{R}^+}, \quad (45)$$

where

$$\sum_{s=1}^n \rho_s(\Gamma^i) = (\Gamma^i \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes \Gamma^i)_{\text{sym}}, \quad (46)$$

where sym stands for the completely symmetrised  $n$ -fold tensor product representation of  $spin(4)$ . Here  $P_{\pm} = \frac{1}{2}(1 \pm \Gamma_5)$ , and  $\mathcal{P}_{\mathcal{R}^+}, \mathcal{P}_{\mathcal{R}^-}$  are projection operators onto the irreducible representations  $\mathcal{R}^+, \mathcal{R}^-$  respectively of  $spin(4)$ . The matrix  $G_5$  which is important to the construction is given by

$$G_5 = \mathcal{P}_{\mathcal{R}^+} - \mathcal{P}_{\mathcal{R}^-}. \quad (47)$$

(Some intuition can be gained from the fact that for  $n = 1$ , the matrices  $G^i$  and  $G_5$  become  $\Gamma^i$  and  $\Gamma_5$  respectively.)

The  $G^i$  are elements of  $End(\mathcal{R})$ . We can write  $G^i = G_+^i + G_-^i$  with  $G_{\pm}^i = \frac{1}{2}(1 \pm G_5)G^i$  and then  $G_{\pm}^i$  act as homomorphisms from  $\mathcal{R}_{\mp}$  to  $\mathcal{R}_{\pm}$ .

From the above definitions, after much manipulation [9] we can obtain the equation

$$G^i + \frac{n+3}{8(2n+1)} \epsilon_{ijkl} G_5 G^j G^k G^l = 0. \quad (48)$$

Thus the Basu-Harvey equation (10) is solved by (12) in the large  $N$  limit. However a solution can be found for any  $n$  by taking

$$X^i(s) = \frac{i\sqrt{2\pi}}{M_{11}^{3/2}} \left( \frac{2n+6}{2n+1} \right)^{1/2} \frac{1}{\sqrt{s}} G^i. \quad (49)$$

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